Level dynamics and the ten-fold way

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Abstract

We investigate the parameter dynamics of eigenvalues of Hamiltonians ('level dynamics') defined on symmetric spaces relevant for condensed matter and particle physics. In particular we: 1) identify appropriate reduced manifold on which the motion takes place, 2) identify the correct Poisson structure ensuring the Hamiltonian character of the reduced dynamics, 3) determine the canonical measure on the reduced space, 4) calculate the resulting eigenvalue density.

1 Introduction

The concept of statistical approach to parametric level dynamics proved to be very fruitful in explaining the applicability of the Random Matrix Theory to the statistics of spectra of generic quantum systems [Haa00]. In the most straightforward setting it consists in considering the flow in a (sub)space Q of $N \times N$ Hermitian matrices

$$X \mapsto X + tY, \tag{1}$$

where Y is a constant, Hermitian, $N \times N$ matrix and t is a real parameter. The matrix $X_t = X + tY$ represents here the Hamiltonian of the quantum system in question, where X and Y describe the 'unperturbed' and 'perturbing' parts, respectively, and t - a coupling parameter controlling the strength of the perturbation. Depending on the symmetries of the investigated system [Haa00], X and Y are general, complex Hermitian matrices, real symmetric matrices, or Hermitian matrices fulfilling $X = JX^tJ^{-1}$ where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2}$$

To gain information about various statistical properties of the distribution of eigenvalues of X_t we should be able to deduct from (1) the parametric motion of

them. This is achieved by treating Q as a configuration space of a Hamiltonian motion in the phase space $Q \times Q$ with the fictitious time t and reducing the dynamics to a smaller manifold which on which the motion is still Hamiltonian and the eigenvalues of X_t are explicitly used as coordinates. The resulting equations can be interpreted as describing the dynamics of a fictitious gas of interacting particles to which one applies rules of statistical mechanics, in. particular, in search of the 'equilibrium' distribution of the particle positions (ie., in fact, the eigenvalues of X_t).

The general reduction procedure was explained in [HZKH01]. Recently it became obvious that besides the above enumerated symmetry classes of Hamiltonians there are other ones, relevant for condensed matter and particle physics [Zir96, HHZ05], where Q is a symmetric space. It is thus of considerable interest to extend the investigations of the parametric level dynamics to these cases. To achieve the goal we should 1) identify appropriate reduced manifold on which the motion takes place, 2) identify the correct Poisson structure ensuring the Hamiltonian character of the reduced dynamics, 3) determine the canonical measure on the reduced space, 4) calculate the resulting eigenvalue density. The above enumerated partial goals will be completed in the consecutive section of the paper.

Let us start with a general description of the setting, and let G/K be one of the following symmetric spaces of non-compact type

$SU(m,n)/S(U(m) \times U(n))$,

$SO(m,n)^0/(SO(m) \times SO(n))$,

$Sp(2m,2n)/Sp(m) \times Sp(n)$,

$SL(n,\mathbb{R})/SO(n)$,

$SL(n,\mathbb{H})/Sp(n)$,

$SO^*(2n)/U(n)$,

$Sp(n,\mathbb{R})/U(n)$.

The configuration space of the considered dynamics of the type (1) is then identified with one of the above.

Note that in every of the above cases there exists a closed embedding of G_0 into $SL_N(\mathbb{C})$ for some $N \in \mathbb{N}$ such that the image is a closed subgroup of $SL_n(\mathbb{C})$ which is closed under conjugate transpose inverse, given as the common zero set of some set of real-valued polynomials in the real and imaginary parts of the matrix entries. In the following we will only consider this image in $SL_N(\mathbb{C})$ which we also denote by G_0. Then K is the fixed point set of the Cartan involution, which is given by $g \mapsto (g^\dagger)^{-1}$ and therefore a subgroup of $SU(N)$.

Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of G_0 and K, respectively. The Cartan decomposition of $\mathfrak{g}_0$ is given by $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ where $\mathfrak{k}_0$ is the $+1$-eigenspace of the Cartan involution $X \mapsto -X^\dagger$ and $\mathfrak{p}_0$ the corresponding -1 eigenspace. Then the symmetric space G_0/K can be identified with $\mathfrak{p}_0$. The corresponding phase space is then the cotangent bundle of G_0/K and can be identified with $\mathfrak{p}_0 \times \mathfrak{p}_0^*$. Using the Killing form on $\mathfrak{g}_0$ which is given by $(X,Y) \mapsto \mathrm{Re}(\mathrm{tr}(XY))$ we can further identify the cotangent bundle with $\mathfrak{p} \times \mathfrak{p}$. The standard symplectic structure of the cotangent bundle then induces the symplectic structure $\omega = \mathrm{Re}(\mathrm{tr}(\mathrm{d}X \wedge \mathrm{d}Y)$ on $\mathfrak{p} \times \mathfrak{p}$.

Let now $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$ and define $S := \mathfrak{a} \times \mathfrak{p} = \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a}^{\perp}$, where $\mathfrak{a}^{\perp}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$ with respect to the inner product $B|_{\mathfrak{p}\times\mathfrak{p}}$. Let (q, p, r) denote the linear coordinates on S corresponding to the decomposition $S = \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a}^{\perp}$. Since $\mathfrak{p} = K \bullet \mathfrak{a}$ the action map $\alpha : K \times S \to \mathfrak{p}\times\mathfrak{p}$, $(k, \xi) \mapsto k \bullet \xi$ is surjective and S is a slice for the K action on N. The stabilizer of K in a generic point $x_0 \in S$ is the centralizer M of $\mathfrak{a}$ in K, $M = Z_K(\mathfrak{a})$. Since M need not to be trivial, S is in general not an exact slice for the K-action. To define an exact slice we need an exact slice for the M-action on $\mathfrak{a}^{\perp}$.

For the last four symmetric spaces the group M is trivial, so $S = \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a}^{\perp}$ is an exact slice, whereas for the first three ones the group M is non trivial and even non-abelian. In the following we compute explicitly the exact slice for the case of the symmetric space $SU(p, q)/S(U(p) \times U(q))$ and then we show how this allows to determine the exact slices for the other two nontrivial cases of $SO(m, n)^0/S(O(m)\times O(n))$ and $Sp(2m, 2n)/Sp(m) \times Sp(n)$.

2 Computing the exact slice

The Cartan decomposition of $\mathfrak{g}_1 = \mathfrak{su}(m, n), m \geq n$, is given by $k_1 \oplus \mathfrak{p}_1$ with

$$\mathfrak{k}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in \mathfrak{u}(m), D \in \mathfrak{u}(n) \text{ and } tr(A) + tr(D) = 0 \right\} = \mathfrak{s}(\mathfrak{u}(m) \times \mathfrak{u}(n)),$$

and

$$\mathfrak{p}_1 = \left\{ \begin{pmatrix} 0 & B \\ \bar{B}^T & 0 \end{pmatrix} \mid B \in M_{m\times n}(\mathbb{C}) \right\}.$$

As a maximal abelian subspace of $\mathfrak{p}_1$ we take

$$\mathfrak{a}_1 = \left\{ X = \begin{pmatrix} 0 & B \\ \bar{B}^T & 0 \end{pmatrix} \text{ with } B = \begin{pmatrix} \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & a_n \\ 0 & \cdot & 0 \\ a_1 & \cdots & 0 \end{pmatrix},\ a_j \in \mathbb{R} \right\}.$$

Then we can compute the centralizer of $\mathfrak{a}_1$ in $K_1 = S(U(m) \times U(n))$. It is given by

$$M_1 := Z_{K_1}(\mathfrak{a}_1) = \left\{ \begin{pmatrix} U & 0 \\ 0 & T \end{pmatrix} \in K_1 \mid U \in U(m-n) \right.$$

and

$$\left. T = diag(e^{it_1}, \ldots, e^{it_n}, e^{it_n}, \ldots, e^{it_1}) \right\}.$$

The restricted roots with respect to $\mathfrak{a}_1$ are given by $\pm 2f_i$, $\pm f_i$ and $\pm f_i \pm f_j$ for $i \neq j$, where $f_i \in \mathfrak{a}_1^*$ is given by

$$f_i : \mathfrak{a}_1 \to \mathbb{R}, \quad X \mapsto a_i.$$

We choose a notion of positivity on this restricted roots such that $f_i, 2f_i, f_i + f_j$ and $f_i - f_j$ with $i < j$ are positive. The corresponding restricted root spaces are given in [Kna05] p. 371 Example 2. The dimensions of the restricted root spaces read

$$\dim \mathfrak{g}_{2f_i} = 1, \quad \dim \mathfrak{g}_{f_i} = m-n \quad \text{and} \quad \dim \mathfrak{g}_{f_i \pm f_j} = 2.$$

Consider the map

$$(1-\theta) : \mathfrak{g}_1 \to \mathfrak{p}_1, \quad X \mapsto X - \theta(X),$$

where θ is the Cartan involution for $\mathfrak{g}_1^{\mathbb{C}} = \mathfrak{sl}(m+n, \mathbb{C})$, so $\theta(X) = -\bar{X}^T$. In particular every restricted root space $\mathfrak{g}_\alpha$ provides a subspace $(1-\theta)(\mathfrak{g}_\alpha)$ of $\mathfrak{p}_1$. The group M_1 acts on each of this spaces and we compute the slice for the action of M_1 on $\mathfrak{a}_1^\perp$ by analyzing the action of M_1 on $(1-\theta)(\mathfrak{g}_\alpha)$.

We start with the restricted roots f_i. The space $(1-\theta)(\mathfrak{g}_{f_i})$ is of the form

$$\left\{ \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ \bar{R}^T & 0 & 0 \end{pmatrix} \middle| \; R = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 & v_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & v_{m-n} & 0 & \cdots & 0 \end{pmatrix} \\ \\ \begin{matrix} & \uparrow & \\ & i & \end{matrix} \end{array} \in M_{m-n,n}(\mathbb{C}) \right\} \simeq \mathbb{C}^{m-n}.$$

The action of M_1 on $(1-\theta)(\mathfrak{g}_{f_i})$ is given by the standard representation of

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \in M_1 \right\} \simeq U(m-n)$$

on $\mathbb{C}^{m-n}$. We will see later that this subgroup acts trivially on the images of the other root spaces, so we can compute the slice separately. The slice for the standard representation of $U(m-n)$ on

$$\bigoplus_{i=1}^{n} (1-\theta)(\mathfrak{g}_{f_i}) \simeq \underbrace{\mathbb{C}^{m-n} \times \cdots \times \mathbb{C}^{m-n}}_{n-times}$$

is given by

$$\left\{ \left(\begin{pmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ c_2 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} * \\ \vdots \\ c_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ \vdots \\ * \\ c_n \end{pmatrix}, \begin{pmatrix} * \\ \vdots \\ * \\ * \end{pmatrix}, \ldots, \begin{pmatrix} * \\ \vdots \\ * \\ * \end{pmatrix} \right) \middle| \; c_j \in \mathbb{R}^+ \right\}.$$

The group M_1 acts trivially on the spaces $(1-\theta)(\mathfrak{g}_{2f_i})$ which are of the form

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & \bar{T}^T & 0 \end{pmatrix} \middle| \; T = antidiag(0, \ldots 0, ia_i, 0, \ldots, 0) \in M_{n,n}(\mathbb{C}) \right\}.$$

The spaces $(1-\theta)(\mathfrak{g}_{f_i \pm f_j})$ are of the form

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & \bar{S}^T & 0 \end{pmatrix} \middle| \; S \in M_{n,n}(\mathbb{C}) \text{ with } s_{ij} = \pm\, \bar{s}_{ji} \in \mathbb{C} \right\} \simeq \mathbb{C}.$$

The group M_1 acts on $(1-\theta)(\mathfrak{g}_{f_i \pm f_j}) \simeq \mathbb{C}$ by $z \mapsto e^{it_i - it_j} z$, so the slice for this action is $\mathbb{R}^+$. Actually we can get this slice for the root spaces corresponding to

simple roots which are given by $f_i \pm f_{i+1}$. Hence the slice for M_1 on $\mathfrak{a}_1^{\perp}$ consists of all matrices of the form

$$
\left(\begin{array}{cc|c}
\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} &
\begin{matrix} 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 \end{matrix} &
\begin{matrix} \diamond & * & * & * & \cdots & * \\ & \ddots & * & * & \cdots & * \\ 0 & & \diamond & * & \cdots & * \end{matrix} \\
\cline{3-3}
\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} &
\begin{matrix} 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 \end{matrix} &
\begin{matrix} * & * & * & * & \diamond & ia_1 \\ * & * & * & \diamond & \cdot & \diamond \\ * & * & \diamond & \cdot & \diamond & * \\ * & \diamond & \cdot & \diamond & * & * \\ \diamond & \cdot & \diamond & * & * & * \\ ia_n & \diamond & * & * & * & * \end{matrix} \\
\hline
\begin{matrix} \diamond & 0 & 0 \\ * & \ddots & 0 \\ * & * & \diamond \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \end{matrix} &
\begin{matrix} * & * & * & * & \diamond & -ia_1 \\ * & * & * & \diamond & \cdot & \diamond \\ * & * & \diamond & \cdot & \diamond & * \\ * & \diamond & \cdot & \diamond & * & * \\ \diamond & \cdot & \diamond & * & * & * \\ -ia_n & \diamond & * & * & * & * \end{matrix} &
\begin{matrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{matrix}
\end{array}\right),
$$

where $*$ are arbitrary elements in $\mathbb{C}$ and $\diamond$ are arbitrary elements in $\mathbb{R}^+$. Lets call this slice $\mathfrak{s}$.

3 The exact slice in the other cases (lining up with $\mathfrak{su}(m,n)$)

In the analysis of other two non-trivial cases in which M is nontrivial we may exploit the results obtained in the previous section by embedding appropriately $\mathfrak{so}(m,n)$ and $\mathfrak{sp}(2m,2n)$ in, respectively, $\mathfrak{su}(m,n)$ and $\mathfrak{su}(2m,2n)$ (lining up with $\mathfrak{su}(m,n)$).

In the case $G_0 = SO(m,n)^0$ we have $\mathfrak{so}(m,n) = \mathfrak{k}_2 \oplus \mathfrak{p}_2$, where

$$\mathfrak{k}_2 = \mathfrak{so}(m,n) \cap \mathfrak{k}_1 = \mathfrak{s}(\mathfrak{o}(m) \times \mathfrak{o}(n))$$

and

$$\mathfrak{p}_2 = \mathfrak{so}(m,n) \cap \mathfrak{p}_1 = \left\{ \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} | B \in M_{m\times n}(\mathbb{R}) \right\}.$$

As a maximal abelian subspace of $\mathfrak{p}_2$ we can choose the same as for $\mathfrak{p}_1$. In particular, we have

$$M_2 = Z_{K_2}(\mathfrak{a}) = M_1 \cap K_2 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} | A \in SO(m-n) \right\} \simeq SO(m-n).$$

Thus in this case the slice is given by matrices of the form

$$
\left(\begin{array}{cc|c}
\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 \end{matrix} & \begin{matrix} * & \cdots & \cdots & \cdots & * \\ & \ddots & & & \vdots \\ 0 & & * & \cdots & * \end{matrix} \\
\hline
\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 \end{matrix} & \begin{matrix} * & \cdots & \cdots & * & 0 \\ \vdots & \ddots & * & \cdot & * \\ \vdots & * & \cdot & * & \vdots \\ * & \cdot & * & & \vdots \\ 0 & * & \cdots & \cdots & * \end{matrix} \\
\hline
\begin{matrix} * & & 0 \\ \vdots & \ddots & \\ \vdots & & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & \cdots & * & 0 \\ \vdots & \ddots & * & \cdot & * \\ \vdots & * & \cdot & * & \vdots \\ * & \cdot & * & & \vdots \\ 0 & * & \cdots & \cdots & * \end{matrix} & \begin{matrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{matrix}
\end{array}\right),
$$

where $*$ are arbitrary elements in $\mathbb{R}$.

In an analogous way we compute the slice for the group $Sp(2m, 2n)$. Obviously we have to do the lining up with $\mathfrak{su}(2m, 2n)$ which we also call $\mathfrak{g}_1$. We also use the same notation for the Cartan decomposition and the maximal Abelian subspaces as for $\mathfrak{su}(m, n)$. The Cartan decomposition of $\mathfrak{g}_3 = \mathfrak{sp}(2m, 2n)$ is given by $\mathfrak{g}_3 = \mathfrak{k}_3 \oplus \mathfrak{p}_3$, where

$$\mathfrak{k}_3 = \mathfrak{g}_3 \cap \mathfrak{k}_1 = \mathfrak{usp}_m \oplus \mathfrak{usp}_n$$

and

$$\mathfrak{p}_3 = \mathfrak{g}_3 \cap \mathfrak{p}_1 = \left\{ \begin{pmatrix} 0 & B \\ \bar{B}^T & 0 \end{pmatrix} \in \mathfrak{g}_3 \text{ with } B \in M_{2m \times 2n}(\mathbb{C}) \right\}.$$

Unfortunately the maximal abelian subspace $\mathfrak{a}_1$ of $\mathfrak{p}_1$ does not contain a maximal abelian subspace of $\mathfrak{p}_3$. Therefore we choose the following maximal abelian subspace $\mathfrak{a}_1'$ of $\mathfrak{p}_1$,

$$\mathfrak{a}_1' := \left\{ X = \begin{pmatrix} 0 & B \\ \bar{B}^T & 0 \end{pmatrix} \text{ with } B = \begin{pmatrix} \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & a_{2n} \\ 0 & \cdot & 0 \\ a_1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \end{pmatrix} \in M_{2m \times 2n}(\mathbb{C}),\ a_j \in \mathbb{R} \right\},$$

where a_{2n} is contained in the $(m-n+1)$th row of the matrix B. Then $M_1' = Z_{K_1}(\mathfrak{a}_1')$ consits of all matrices of the form

$$\begin{pmatrix} U_1 & 0 & U_2 & 0 \\ 0 & D_1 & 0 & 0 \\ U_3 & 0 & U_4 & 0 \\ 0 & 0 & 0 & D_2 \end{pmatrix} \in S(U(m) \times U(n)) \text{ with } \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in U(2m - 2n),$$

$D_1 = diag(e^{it_1}, ..., e^{it_{2n}})$ and $D_2 = diag(e^{it_{2n}}, ..., e^{it_1})$. Now one can choose a maximal abelian subspace $\mathfrak{a}_3$ in $\mathfrak{p}_3$ which is contained in $\mathfrak{a}_1'$, namely

$$\mathfrak{a}_3 = \left\{ X = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathfrak{a}_1' \text{ with } a_i = a_{i+n} \in \mathbb{R}, 1 \le i \le n \right\},$$

and then $M_3 = M_1' \cap K_3$ consists of all matrices of the form

$$\begin{pmatrix} U_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \\ 0 & 0 & \bar{U}_1 & 0 \\ 0 & 0 & 0 & D_2 \end{pmatrix} \in USp(m) \times USp(n) \text{ with } U_1 \in U(m-n),$$

$D_1 = diag(e^{it_1}, ..., e^{it_n}, e^{it_1}, ..., e^{it_n})$ and $D_2 = diag(e^{it_n}, ..., e^{it_1}, e^{it_n}, ..., e^{it_1})$.

4 Poisson Structure

We use the exact slice to compute the Poisson structure with respect to the new coordinates which are given as follows. Let $Q = (q_1, ..., q_N)$ denote the coordinates for $\mathfrak{a}$ in the first factor of the product $S = \mathfrak{a} \times \mathfrak{a} \oplus \mathfrak{s}$ regarded as an K-invariant function on $\mathfrak{p} \times \mathfrak{p}$. Further let V denote the standard matrix coordinates of the second factor of S again regarded as an invariant function on $\mathfrak{p} \times \mathfrak{p}$. Finally for $x = k \cdot s$ with $s \in S$ and $x \in \mathfrak{p} \times \mathfrak{p}$ we define $U(x) \in K$ to be the matrix given by k. Define $\mathrm{d}U$ to be the matrix of 1-forms $\mathrm{d}U_{ij}$ and $W := U^{-1}\mathrm{d}U$. Note that $W = -W^\dagger$ since $U^\dagger U = Id$. This also gives $\mathrm{d}W = -W \wedge W$.

We now compute the symplectic form $\omega = \mathrm{Re}(\mathrm{tr}(\mathrm{d}X \wedge \mathrm{d}Y)$ using these invariant functions. For this we write $\omega = \mathrm{d}\theta$, where $\theta = \mathrm{Re}(\mathrm{tr}(Y\mathrm{d}X))$. We have thus

$$\begin{aligned}
\theta &= \mathrm{Re}(\mathrm{tr}(UVU^{-1}\mathrm{d}(UQU^{-1}))) \\
&= \mathrm{Re}(\mathrm{tr}(UVU^{-1}\mathrm{d}UQU^{-1} + \mathrm{Re}(\mathrm{tr}(UVU^{-1}U\mathrm{d}QU^{-1})) + \mathrm{Re}(\mathrm{tr}(UVU^{-1}UQ\mathrm{d}U^{-1})) \\
&= \mathrm{Re}(\mathrm{tr}(VWQ)) + \mathrm{Re}(\mathrm{tr}(V\mathrm{d}Q)) + \mathrm{Re}(\mathrm{tr}(VQW^{-1}) \\
&= -\mathrm{Re}(\mathrm{tr}(VW^{-1}Q)) + \mathrm{Re}(\mathrm{tr}(V\mathrm{d}Q)) + \mathrm{Re}(\mathrm{tr}(VQW^{-1}) \\
&= \mathrm{Re}(\mathrm{tr}(V\mathrm{d}Q)) + \mathrm{Re}(\mathrm{tr}(V[Q, W^{-1}]))).
\end{aligned}$$

We can now simplify the second summand,

$$\begin{aligned}
\mathrm{Re}(\mathrm{tr}(V[Q, W^{-1}]))) &= -\mathrm{Re}(\mathrm{tr}([V, Q]W)) \\
&= -\frac{1}{2}\mathrm{tr}([V, Q]W + \overline{[V, Q]}\bar{W}) \\
&= -\frac{1}{2}\mathrm{tr}([V, Q]W + [V, Q]^T W^T) \\
&= -\mathrm{tr}([V, Q]W) \\
&= -\mathrm{tr}(lW),
\end{aligned}$$

where we again use $\mathfrak{k}_0 \subset \mathfrak{su}(N)$. Therefore we have

$$\omega = \mathrm{d}\theta = \mathrm{Re}(\mathrm{tr}(\mathrm{d}V \wedge \mathrm{d}Q)) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W).$$

Due to the structure of $\mathfrak{s}$ we can replace $\mathrm{d}V$ by its $\mathfrak{a}$-part with respect to the decomposition $\mathfrak{a} \oplus \mathfrak{s}$ which we call $\mathrm{d}P$. Moreover, since Q and P are real we have

$$\omega = \mathrm{tr}(\mathrm{d}P \wedge \mathrm{d}Q) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W),$$

which shows that the Poisson structure splits with the pair (P, Q) being canonical, commuting with l. We want show now that l has the coadjoint Poisson structure of $\mathfrak{k}_0$.

Proposition 4.1. *The map $l : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{k}^*$ is a Poisson morphism.*

We regard l as a complex matrix valued map with values in $\mathfrak{s}(\mathfrak{u}(m) \times \mathfrak{u}(n))$ and let $\mathrm{d}l = (\mathrm{d}l_{ij})$ be the matrix of $\mathbb{C}$-valued 1-forms.

We now compute the Hamilton vector field V_f of a function $f = f(l)$ of l alone. It is defined by the equation $\mathrm{d}f(Z) = \omega(V_f, Z)$ for any real valued field Z on $\mathfrak{p} \times \mathfrak{p}$. Since the symplectic forms splits an f is a function of l alone we only have to consider the pieces of the field Z which involve $(\partial/\partial l)$ and $(\partial/\partial W)$. Therefore we have

$$\omega(V_f, Z) = -\mathrm{tr}(\mathrm{d}l \wedge W - lW \wedge W)(A, B),$$

where $A = \mathrm{tr}(V_f^l(\partial/\partial l)^T) + \mathrm{tr}(V_f^W(\partial/\partial W)^T)$ and $B = \mathrm{tr}(Z^l(\partial/\partial l)^T) + \mathrm{tr}(Z^W(\partial/\partial W)^T)$. Then we have

$$\begin{aligned} \omega(V_f, Z) &= -\mathrm{tr}(V_f^l Z^W + Z^l V_f^W + l V_f^W Z^W - l Z^W V_f^W) \\ &= \mathrm{tr}(V_f^l Z^W + Z^l V_f^W + [l, V_f^W] Z^W). \end{aligned}$$

This implies

$$\mathrm{d}f(Z) = \mathrm{tr}(Z^l(\partial f/\partial l)^T = \mathrm{tr}(Z^l V_f^W) + \mathrm{tr}(Z^W(-V_f^l + [l, V_f^W]),$$

and, therefore, $V_f^W = (\partial f/\partial l)^T$ and $V_f^l = [l, (\partial f/\partial l)^T]$.

5 The cotangent bundle

The moment map on the cotangent bundle $N := T^*Q$ is given by

$$\mu \colon T^*Q \to \mathfrak{k}^*, \quad \alpha \mapsto (\xi \to \alpha(\xi_X(\pi(\alpha))).$$

After identifying T^*Q with $\mathfrak{p} \times \mathfrak{p}$ we have

$$\mu(X_1, X_2)(\xi) = B(X_2, [\xi, X_1]),$$

where B is as above. Using the K-invariance of B we get

$$\mu(X_1, X_2)(\xi) = B(X_2, [\xi, X_1]) = -B(X_2, [X_1, \xi]) = B([X_1, X_2], \xi).$$

With the identification $\mathfrak{k}^* \simeq \mathfrak{k}$ by the inner product $-B|_{\mathfrak{k} \times \mathfrak{k}}$ this gives

$$\mu(X_1, X_2) = [X_2, X_1].$$

Define $l \colon S \to \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^{\perp_{\mathfrak{k}}}$ by $(q, p, r) \mapsto \mu(q, p, r) = [r, q]$. Since $\mathfrak{p} = K \bullet \mathfrak{a}$ and $Z_K(\mathfrak{a})$ fixes $\mathfrak{a}$ pointwise the map

$$f \colon K/Z_K(\mathfrak{a}) \times \mathfrak{a} \to \mathfrak{p}, ([k], \xi) \mapsto \mathrm{Ad}(k)\xi,$$

is well defined and surjective. Let E be a generic point of $\mathfrak{a}$, e. g. E is the half sum of positive roots. Since the stabilizer of K in E is precisely $Z_K(\mathfrak{a})$ the derivative of f in the point $([e], E)$, which is given by

$$\begin{aligned} Df([e],E) : T_{([e],E)}(K/Z_K(\mathfrak{a}) \times \mathfrak{a}) \simeq \mathfrak{k}/\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \times \mathfrak{a} \to \mathfrak{p}, \\ ([\xi], \eta) \quad \mapsto \xi_X(E) + \eta = [\xi, E] + \eta, \end{aligned}$$

is an isomorphism of $Z_K(\mathfrak{a})$-representations. Using the inner product $-B|_{\mathfrak{k}\times\mathfrak{k}}$ we can identify $\mathfrak{k}/\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ with $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$ and we get an isomorphism of the $Z_K(\mathfrak{a})$-representation spaces $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$ and $\mathfrak{a}^\perp$ given by $\xi \mapsto [\xi, E]$.

Remark 5.1. *Let $q \in \mathfrak{a}$. Then the linear map $A_q := \mathrm{ad}(E) \circ \mathrm{ad}(q) \colon \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp \to \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$ is selfadjoint since*

$$\kappa(A_q(\xi), \eta) = \kappa([E, [q, \xi], \eta]) = \kappa(\xi, [q, [E, \eta]]) = \kappa(\xi, [E, [q, \xi]]) = \kappa(\xi, A_q(\eta))$$

for $\xi, \eta \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$. In particular, A_q is diagonalizable.

Let e_j be a basis for $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$ consisting of eigenvectors of A_q with eigenvalues $\lambda_j(q)$. The same is true if we regard A_q as a map from $\mathfrak{a}^\perp$ to itself.

6 Slice densities

Let $\mathrm{d}\lambda_M$ denote the Liouville measure associated with the Liouville form $\omega_N := \omega^n$ on N. Further let ω_S denote the linear volume form on the slice S. Then the slice density $\rho : S \to \mathbb{R}^{\geq 0}$ is given by the equation

$$\int_N f \ \omega_N = \int_S (f \cdot \rho) \ \omega_S$$

for all compactly supported functions $f \in \mathcal{E}_0(M)^K$. Our goal is to compute this slice density and to prove the following.

Proposition 6.1. *The canonical slice measure is given by $\rho \ \mathrm{d}\lambda_S = \mathrm{d}q \ \mathrm{d}p \ \mathrm{d}l$*

Let $\alpha \colon K \times S \to M, (k, x) \mapsto k \bullet x$ denote the action map. This is a $Z_K(\mathfrak{a})$-principal bundle and we can compute the slice density as follows.

Let ω_K the invariant volume form on K normalized by $\int_K \omega_K = 1$ and let ω_S the standard Euclidean volume form $\mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}r$ on S. Let $\mathcal{T}$ be the invariant frame field along the fibers of α. Then we define the function $\rho\colon K \times S \to \mathbb{R}^{\geq 0}$ by the equation

$$\rho \ \iota_{\mathcal{T}}(\omega_K \wedge \omega_S) = \alpha^* \omega_M$$

where $\iota_{\mathcal{T}}$ denotes contraction with the frame $\mathcal{T}$. Since all of the differential forms which are involved are invariant under the group K, the function ρ is also K-invariant and therefore defines a function on the slice S which we also denote by ρ.

Applying Fubini's Theorem we get

$$\int_{K \times S} \alpha^*(f) \ \rho \ \omega_K \wedge \omega_S = \int_S \Big(\int_K \rho \alpha^*(f) \ \omega_K \Big) \omega_S = \int_S f \rho \ \omega_S$$

for any $f \in \mathcal{E}_0(M)^K$. Let $\omega_{Z_K(\mathfrak{a})}$ denote the invariant volume form on $Z_K(\mathfrak{a})$ normalized by $\int_{Z_K(\mathfrak{a})} \omega_{Z_K(\mathfrak{a})} = 1$ such that

$$\omega_{Z_K(\mathfrak{a})} \wedge \iota_{\mathcal{T}}(\omega_K \wedge \omega_S) = \omega_K \wedge \omega_S.$$

Then by fiber integration we get

$$\int_{K \times S} \alpha^*(f) \rho \, \omega_K \wedge \omega_S = \int_{K \times S} \alpha^*(f) \; \rho \; \omega_{Z_K(\mathfrak{a})} \wedge \iota_{\mathcal{T}}(\omega_K \wedge \omega_S)$$

$$= \int_{K \times S} \omega_{Z_K(\mathfrak{a})} \wedge \alpha^*(f \, \omega_M)$$

$$= \int_M \left(\int_{Z_K(\mathfrak{a})} \omega_{Z_K(\mathfrak{a})} \right) f \, \omega_M = \int_M f \, \omega_M.$$

This shows that the function ρ is the slice density defined above. In the following we want to compute ρ in a more explicit way. At $s = (q, p, l) \in S$, we compute the determinant of the projection of the map

$$\alpha_* : \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp \to \mathfrak{p} \times \mathfrak{p} \simeq \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp \times S, \quad \xi \mapsto \alpha_*(\xi),$$

onto the factor $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$. It can be computed as follows.

$$\alpha_*(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha(\exp(t\xi), s) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(t\xi))(s) = [\xi, s] = ([\xi, q], [\xi, (p, l)]),$$

with $[\xi, q] \in \mathfrak{a}^\perp \simeq \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^\perp$. Then we have

$$Pr(\alpha_*(\xi)) = Pr([\xi, s]) = [\xi, q].$$

In particular, we get

$$Pr(\alpha_*(e_j)) = \lambda_j(q) e_j,$$

which gives

$$\rho(s) = \Big| \prod_j \lambda_j(q) \Big|.$$

Proof of Proposition 6.1. Let (q, p, r) denote the linear coordinates on $S = \mathfrak{a} \times \mathfrak{p}$. Then $\omega_S = \mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}r$ is the linear volume form on S. Consider the coordinate change

$$\varphi : S \to S, \quad (q, p, r) \mapsto (q, p, l) := (q, p, \mu(q, r)) = (q, p, [r, q]).$$

By the transformation rule we get

$$\mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}l = |\mathrm{D}\varphi(q, p, r)| \cdot \mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}r = \Big| \prod_j \lambda_j(q) \Big| \cdot \omega_S = \rho \, \omega_S.$$

□

As an example let us compute explicitly the slice density for the symmetric space $SU(m, n)/S(U(m) \times U(n))$

We consider the following basis of $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})^{\perp}$. Let $e^k_{i,j}, k = 1,2; 1 \leq i \leq m-n; 1 \leq j \leq n$ and $i < j$ denote the matrices of the form

$$\begin{pmatrix} 0 & B & 0 \\ -\bar{B}^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } B = (b_{k,l}) \in M_{(m-n)\times n}(\mathbb{C})$$

$$\text{and } b_{k,l} = \begin{cases} 1 & \text{for } (k,l) = (i,j) \text{ and } k = 1 \\ i & \text{for } (k,l) = (i,j) \text{ and } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $A_q \cdot e^k_{i,j} = q_j \cdot e^k_{i,j}$. Further let $e_i, 1 \leq i \leq n$ denote the matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C' \end{pmatrix}, \text{ where } C = diag(c_1, ..., c_n) \text{ with } c_l = \begin{cases} i & \text{for } l = i \\ 0 & \text{otherwise,} \end{cases}$$

and $C' = diag(-c_q, ..., -c_1)$. Then $A_q \cdot e_i = q_i \cdot e_i$. We further define the basis elements $f^{\pm}_{i,j}, 1 \leq i < j \leq n$ to be the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D' \end{pmatrix} \text{ where } D = (d_{kl}) \in M_{n\times n}(\mathbb{C}) \text{ with } d_{kl} = \begin{cases} 1 & \text{for } (k,l) = (i,j) \\ -1 & \text{for } (k,l) = (j,i) \\ 0 & \text{otherwise,} \end{cases}$$

and D' is equal to $\pm D$ reflected at the anti-diagonal. For these basis elements we have $A_q \cdot f^{\pm}_{i,j} = (q_i \pm q_j) f^{\pm}_{i,j}$. The last basis elements $g^{\pm}_{i,j}, 1 \leq i < j \leq n$ are given by matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D' \end{pmatrix} \text{ where } D = (d_{kl}) \in M_{n\times n}(\mathbb{C}) \text{ with } hd_{kl} = \begin{cases} i & \text{for } (k,l) = (i,j) \\ i & \text{for } (k,l) = (j,i) \\ 0 & \text{otherwise,} \end{cases}$$

and D' is again $\pm D$ reflected at the anti-diagonal. We have $A_q \cdot g^{\pm}_{i,j} = (q_i \pm q_j) \cdot g^{\pm}_{i,j}$. Therefore the slice density is given by

$$\rho(s) = \prod_{i=1}^{n} q_i^{2(m-n)+1} \cdot \prod_{i<j} (q_i^2 - q_j^2)^2.$$

By an analogous computation for $SO(m,n)^0/S(O(m) \times O(n))$ we get the slice density

$$\rho(s) = \Big| \prod_{i=1}^{n} q_i^{m-n} \cdot \prod_{i<j} (q_i^2 - q_j^2) \Big|$$

Remark. Note that the density $\rho(s)$ only depends on the variable q, i.e., on the eigenvalues of the operators at hand. When formulated in our notation, the usual procedure in random matrix theory is to start with a K–invariant probability density d on the cotangent bundle of the symmetric space so that the resulting density on the slice defines a probability measure $d\rho dq dp dr$. In classical examples where M is not present it is usually a simple matter to compute the image "spectral" measure on $\mathfrak{a}$ or on the Weyl chamber $\mathfrak{a}_+$. It would be interesting to know if the presence of M is of physical interest, e.g., if it would be appropriate to simply take the standard norm function and use K-invariant probability distribution $\exp(-\frac{1}{2}\| \cdot \|^2)\, d\lambda_M$.

7 Other symmetric spaces

Symmetric spaces of Type II.

Above we restricted our discussion to symmetric spaces of simple Lie groups which are *not* complex. If the real group G happens to be complex, one refers to the associated symmetric space G/K as being of Type II. A typical example is $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n$. Actually the above discussion simplifies in this situation. The point is that if G is complex, then the subgroup K is a compact real form and at the Lie algebra level $\mathfrak{p} = i\mathfrak{k}$. If $\mathfrak{t}$ is the Lie algebra of a maximal torus in $\mathfrak{k}$, then $\mathfrak{a} := i\mathfrak{t}$ is a maximal Abelian subspace of $\mathfrak{p}$. Since the centralizer of $\mathfrak{t}$ in $\mathfrak{k}$ is $\mathfrak{t}$ itself, it follows that $\mathfrak{m} = 0$.

For example, in the case of $G = \mathrm{SL}_n(\mathbb{C})$ the Cartan decomposition of a matrix in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ amounts to writing it as the sum of antihermitian and hermitian matrices. Hence, our work above just puts the classical level dynamics discussion ([Haa00],[HZKH01]) for pairs of hermitian matrices in a symmetric space framework.

Symmetric spaces of compact type

If G is compact and G/K is a symmetric space, e.g., the Grassmannian $\mathrm{Gr}_q(\mathbb{C}^n)$ of q–dimensional complex subspaces in $\mathbb{C}^n$, then one can also discuss level dynamics in a setup similar to that above. Conceptually it is convenient to think about this in a situation where the the duality between symmetric spaces of compact and noncompact type is visible. For this it is convenient to introduce a bit of notation. Details of the below discussion can be found in ([FHW05]).

If G_0 is a simple Lie group of noncompact type with a given Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, we consider the complexification $\mathfrak{g} := \mathfrak{g}_0 + i\mathfrak{g}_0$ and the associated complex semisimple Lie group G. Note that in the Type II case mentioned above $\mathfrak{g}$ is the direct sum of two copies of $\mathfrak{g}_0$. Otherwise, $\mathfrak{g}$ is also simple.

One observes that $\mathfrak{g}_u = \mathfrak{k}_0 + i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g}$. So, going to the Lie group level, we have the complex group G containing the noncompact real form G_0 and the compact real form G_u. Now let K_0 be the maximal compact subgroup of G_0 which is associated to $\mathfrak{k}_0$ and K the complex subgroup of G which is associated to $\mathfrak{k} = \mathfrak{k}_0 + i\mathfrak{k}_0$. If x_0 is the neutral point in the complex (affine) symmetric space G/K, then $G_0.x_0 = G_0/K_0$ is initial symmetric space of noncompact type and $G_u.x_0 = G_u/K_0$ is the dual symmetric space of compact type. The cotangent space at the neutral point of the noncompact symmetric space is $\mathfrak{p}_0^*$ and that of the compact symmetric space is $i\mathfrak{p}_0^*$.

Above in the case of noncompact symmetric spaces we have use the fact that using the exponential map we may identify the given symmetric space with $\mathfrak{p}_0$. In particular the cotangent bundle is trivial. This is essentially never the case for compact symmetric spaces, e.g., almost no spheres have this property. Furthermore, the exponential map $\exp : i\mathfrak{p}_0 \to G$ is not as simple in this case. The difficulty can, however, be isolated in the maximal Abelian subspace $i\mathfrak{a}_0$ whose associated group is a compact torus. Here $\exp : i\mathfrak{a}_0 \to A$ is nothing other than the usual covering mapping which amounts to dividing out a vector space by a lattice of periods.

Using this and the fact that (modulo a certain Weyl group) $i\mathfrak{a}_0$ is a slice for the K_0–action on $\mathfrak{p}_0$, one observes that, after going to the complement $i(\mathfrak{p}_0)_{\mathrm{gen}}$ of an approriate set of measure zero in $\mathfrak{p}_0$, we have an identification of the phase space at

hand with the product $i(\mathfrak{p}_0)_{\mathrm{gen}} \times i\mathfrak{p}_0^*$. All of the above considerations for symmetric spaces of noncompact type can now be carried out on this set of *generic points* in the cotangent bundle of the compact symmetric space.

The above indicates that obtaining coordinates for considerations of level dynamics in the cotangent bundle of symmetric space of compact type is a more difficult matter than in the case of noncompact symmetric spaces. On the other hand, the compact symmetric space has one major advantage: the complex symmetric space G/K is naturally identifiable with its cotangent bundle. In other words, the relevant phase space is itself a complex symmetric space. This is not the case for the noncompact symmetric space. There is indeed a map from its cotangent bundle into G/K (polar coordinates), but this degenerates at a certain point. There is, however, a precisely defined maximal neighborhood of the zero–section of this phase where the polar coordinate mapping is a diffeomorphism onto its image $\mathcal{U}$ in G/K. Thus on $\mathcal{U}$, where perfect coordinates and natural invariant measures are available, it is possible to consider the level dynamics related to both the compact and noncompact symmetric spaces. .

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